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# BOUNDARY LAYERS ON ROTATING SPHERES AND OTHER AXISYMMETRIC SHAPES

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## SUMMARY

Most of this report is concerned with the laminar boundary layer on a rotating sphere in a fluid at rest. The existing controversy as to the behavior of the boundary layer near the equator is examined in the light of the governing equations. Errors were found in one group of previous solutions. A new series solution that agrees with existing experimental results provides a comparison with the consequences of the erroneous solutions. The new solution is so formulated that boundary layers on many axisymmetric shapes with conical tips (including spheres) can be calculated. Numerical values of four orders of series functions are given.

## INTRODUCTION

All investigators of the boundary layer on a rotating sphere in a quiescent fluid agree that the flow near the poles is similar to that on a rotating disk. The rotating surface swirls the fluid around, and the centrifugal forces induce a flow away from the axis, which is accompanied by an inflow to the boundary layer from the surrounding fluid. The flow behavior near the equator is in dispute.

Howarth (ref. 1) developed the boundary-layer equations and proposed a series solution. No calculations were made for the series; instead, a polynomial solution to the momentum integral equations, in which certain terms were neglected, was calculated. Flow through the equatorial plane from both hemispheres was predicted by the solution. Howarth deduced that the actual boundary layers collide at the equator and are flung outward in a manner that cannot be described by the boundary-layer equations.

Nigam (ref. 2) disputed these findings and proposed a countersolution to the same boundary-layer equations which specified that the flow toward the equator slows and stops so that no fluid crosses the equatorial plane. According to Nigam, the fluid leaves the boundary layer by flowing outward away from the sphere between 55° and 90° from the pole with no collision occurring at the equator. Approximate values of the series functions were obtained by a polynomial method. Other investigators have extended Nigam's analysis to spheroids, heat transfer, and non-Newtonian fluids (refs. 3 to 5). Low Reynolds number flows have been treated with somewhat similar assumptions (ref. 6).

Stewartson considered Nigam's statement that the fluid flowed out of the boundary layer before the equator and concluded that such a flow could not exist (ref. 7). An experimental velocity survey (ref. 8) indicated both an outflow before the equator and an equatorial jet that resulted from the collision of the two boundary layers. Recently, Bowden and Lord (ref. 9) measured the torque on a magnetically suspended sphere. In smoke studies, no outflow was noted except that in the equatorial jet (refs. 9 and 10).

In the present review of the problem, Nigam's solution is examined in some detail. The unusual features of the analytic formulation are contrasted with the features of conventional formulations and with the physical implications of the governing momentum equations. A comparison with a new solution shows some of the consequences of Nigam's formulation.

The new series solution is so formulated that boundary layers on a family of spinning shapes with conical tips can be calculated by the use of the given series functions. The calculated torque value for a sphere agrees with an experimental value measured by Bowden and Lord (ref. 9).

## NIGAM'S SOLUTION

### Analytic Formulation

A general agreement exists in the form of the constant-property boundary-layer equations on a rotating sphere

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} - \frac{w^2}{a} \cot\left(\frac{x}{a}\right) = \nu \frac{\partial^2 u}{\partial y^2} \tag{1a}$$

$$u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + \frac{uw}{a} \cos\left(\frac{x}{a}\right) = \nu \frac{\partial^2 w}{\partial y^2} \tag{1b}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{u}{a} \cot\left(\frac{x}{a}\right) = 0 \tag{1c}$$

where $x$ and $u$ are the distance and velocity components along the sphere generator from the pole, $y$ and $v$ are the distance and velocity components normal to the spherical surface, and $z$ and $w$ are the distance and velocity components along the circles formed by the intersection of the sphere with planes normal to the axis. The symbols $a$ and $\nu$ designate the sphere radius and kinematic viscosity.

Derivatives in the z-direction are zero by symmetry. The adherence of the fluid to the spinning sphere produces the only nonzero boundary condition $w_{y=0} = a\omega \sin (x/a)$, where $\omega$ is the rotational speed.

Nigam proposed a solution in the following form:

$$u = a\omega \cos \theta \left[F_1(Y)\sin \theta + F_3(Y)\sin^3\theta + F_5(Y)\sin^5\theta + \ldots\right] \tag{2a}$$

$$v = (\nu\omega)^{1/2}\left(1 - \frac{3 \sin^2\theta}{2}\right)\left[H_1(Y) + H_3(Y)\sin^2\theta + H_5(Y) \sin^4\theta + \ldots\right] \tag{2b}$$

$$w = a\omega\left[G_1(Y)\sin \theta + G_3(Y)\sin^3\theta + G_5(Y)\sin^5 \theta + \ldots\right] \tag{2c}$$

where $Y = y(\omega/\nu)^{1/2}$ and $\theta = x/a$. One nonzero boundary condition $G_1(0) = 1$ is used; the remaining boundary conditions ($F_1(0)$, $F_1(\infty)$, etc., except $H_1(\infty)$, $H_3(\infty)$, . . .) are zero at $Y = 0$ and as $Y$ approaches infinity.

The first-order approximations to $u$, $v$, and $w$, namely $F_1$, $H_1$, and $G_1$, are the well-known similarity results for a rotating disk that also satisfy the full Navier-Stokes equations. As a result, Nigam's solution is well-founded in rigorous fluid mechanics at the pole.

In addition to the initial value of $u = 0$ at the pole, the form of the series in equation (2a) prescribes a zero value for $u$ at the equator ($\theta = \pi/2$), which is contrary to the conventional practice with boundary-layer equations. Usually, just an initial velocity $u(0,y)$ is specified because the boundary layer equations are first-order equations in the x-direction. On a sphere the initial motion at the pole is like that on a rotating disk, where the similarity of the velocity profiles is used instead of an initial condition as such to find the solution to equations (1). At a general point the rate of change of $u$ in the downstream (x-) direction is specified by a convective accelleration term $u(\partial u/\partial x)$ in equation (1a) as a response to local forces. Nigam's solution accommodates changes from the initial similarity solution by means of contributions from the higher order terms of the series. These contributions begin, in effect, at successively greater distances from the pole as the order increases. At the equator, however, the higher order terms cannot correct the solution for nonsimilar conditions because all the terms are zero. This is equivalent to a downstream restriction $u(\pi/2,y) = 0$ on the boundary layer. There are no means in the boundary-layer equations (1) by which this downstream condition can be accomodated. No information can be transmitted upstream by any of the terms in equations (1).

It is evident, however, from the symmetry of the spherical flow field that $u(\pi/2,y) = 0$ is a condition that exists in the actual flow even though the boundary-layer equations cannot accommodate it. The possibility that a valid solution to the boundary-layer equations naturally contains this equatorial condition, with no accommodation, is examined in the next section.

## Momentum Considerations

The terms in equation (1a) are considered herein in an intuitive deduction of the boundary-layer dynamics near the equator. As a first step, the physical roles of the terms in equation (1a) are stated. The first two terms represent the convective acceleration component. Centrifugal and viscous forces are specified by the last two terms. Flow in the x-direction is induced by the component of the centrifugal forces that is parallel to the surface, as specified by the third term in equation (1a). The sense and relative magnitude of this component can be visualized by evaluating equation (1a) at the surface:

$$- a\omega^2 \sin \theta \cos \theta = \nu \left(\frac{\partial^2 u}{\partial y^2}\right)_{y=0} \tag{3}$$

The centrifugal term in equation (3) increases initially with $\theta$, reaches its maximum at $\theta = \pi/4$, and decreases to zero at the equator. On the entire hemisphere this term tends to accelerate the flow. The viscous term in equation (1a) together with the zero boundary conditions $u(\theta,0) = u(\theta,\infty) = 0$ tends to reduce $u$ by momentum diffusion. Nevertheless, it seems evident that some $u > 0$ exists in the fluid that approaches the equator because of the inertia of the fluid and the accelerating influences.

Nigam's equatorial condition $u(\pi/2,y) = 0$, which must also exist in an experiment, is not compatible with boundary-layer dynamics according to the governing equations. This incompatibility implies that the boundary-layer flows collide and flow outward along the equatorial plane, as Howarth visualized (ref. 1) and Bowden and Lord showed experimentally (ref. 9). This behavior is also implied by the new boundary-layer solution that yields $u(\pi/2,y) > 0$.

## NEW SOLUTION

The new series solution describes boundary layers on axisymmetric spinning bodies with conical tips. In this sense a sphere is regarded as a cone-tipped body with a vertex angle of $\pi$. The boundary-layer equations for this case are shown in the form used by Geis (ref. 11):

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} - \frac{w^2 r'}{r} = \nu \frac{\partial^2 u}{\partial y^2} \tag{4a}$$

$$u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + \frac{uwr'}{r} = \nu \frac{\partial^2 w}{\partial y^2} \tag{4b}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{ur'}{r} = 0 \tag{4c}$$

were $r = r(x)$ is the radius of the surface measured normal to the axis, and $r' = dr/dx$. The other symbols are used in the same sense as on a sphere (eqs. 1).

Series expansions of the stream function $\psi$ have been widely used in fluid mechanics but not in previous solutions to this problem. Howarth and Nigam chose to expand the three velocity components in separate series (refs. 1

and 2). Stream function expansions satisfy the continuity equation identically, of course, and thereby reduce the number of dependent variables in equations (4) by one.

If a convenient reference length L is available, the solution is

$$ru = \frac{\partial\psi}{\partial y}, \quad rv = -\frac{\partial\psi}{\partial x} \tag{5a}$$

$$\psi = L^2(\nu\omega r_1)^{1/2}\Big[X^2 r_1 f_1(\eta) + X^4 r_3 f_3(\eta) + X^6\Big(r_5 f_{5a}(\eta) + \frac{r_3^2}{r_1} f_{5b}(\eta)\Big) + X^8\Big(r_7 f_{7a}(\eta) + \frac{r_3 r_5}{r_1} f_{7b}(\eta) + \frac{r_3^3}{r_1^2} f_{7c}(\eta)\Big) + \ldots\Big] \tag{5b}$$

$$w = L\omega\Big[X r_1 g_1(\eta) + X^3 r_3 g_3(\eta) + X^5\Big(r_5 g_{5a}(\eta) + \frac{r_3^2}{r_1} g_{5b}(\eta)\Big) + X^7\Big(r_7 g_{7a}(\eta) + \frac{r_3 r_5}{r_1} g_{7b}(\eta) + \frac{r_3^3}{r_1^2} g_{7c}(\eta)\Big) + \ldots\Big] \tag{5c}$$

$$r = L\Big[X r_1 + X^3 r_3 + X^5 r_5 + \ldots\Big] \tag{5d}$$

$$\eta = y\left(\frac{\omega r_1}{\nu}\right)^{1/2} \tag{5e}$$

where $X = x/L$ is not necessarily associated with an angle. On a sphere, however, $X = \theta$ and $L = a$ seem most convenient. Dimensionless coefficients $r_1, r_2, \ldots$ are used in equations (5) but equivalent dimensional coefficients are easily obtained by eliminating L. The dimensions of powers of L are carried by the coefficients, and X is replaced by x. In either case $r_1 = \sin\gamma$ is dimensionless, where $\gamma$ is one-half the cone vertex angle at the tip.

Boundary conditions are

$$\left.\begin{aligned} f_1 &= f_3 = f_{5a} = f_{5b} = f_{7a} = f_{7b} = f_{7c} = 0 \\ f_1' &= f_3' = f_{5a}' = f_{5b}' = f_{7a}' = f_{7b}' = f_{7c}' = 0 \\ g_1 &= g_3 = g_{5a} = g_{7a} = 1 \\ g_{5b} &= g_{7b} = g_{7c} = 0 \end{aligned}\right\} \eta = 0 \tag{6a}$$

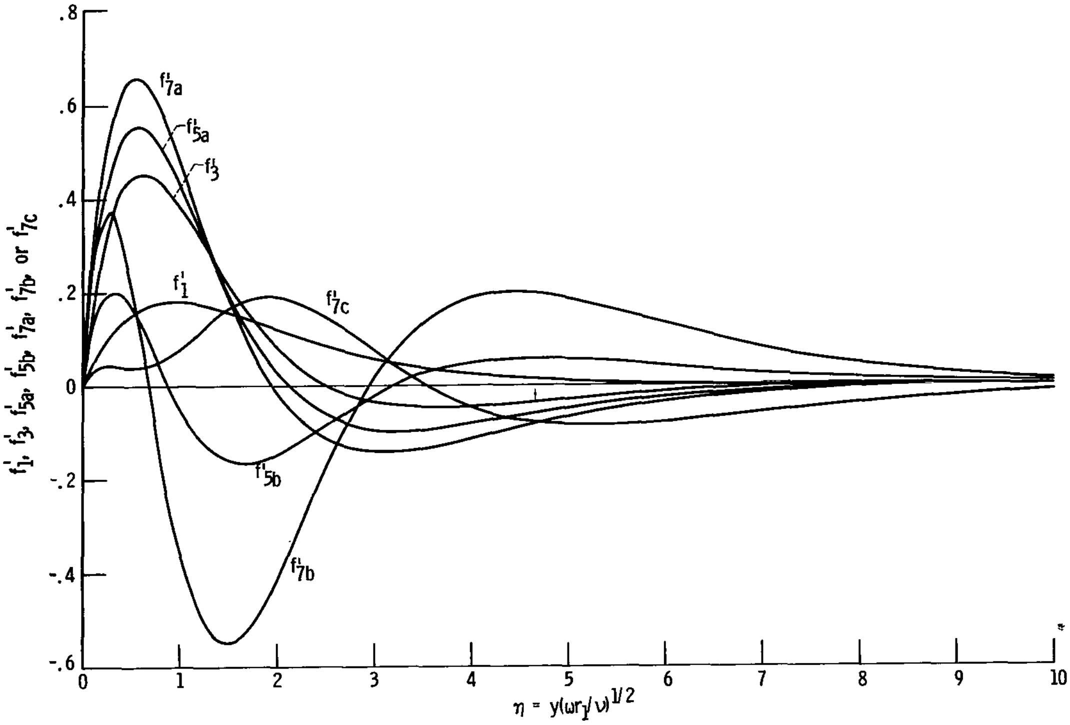


Figure 1. - Series functions related to velocity component in x-direction, u.

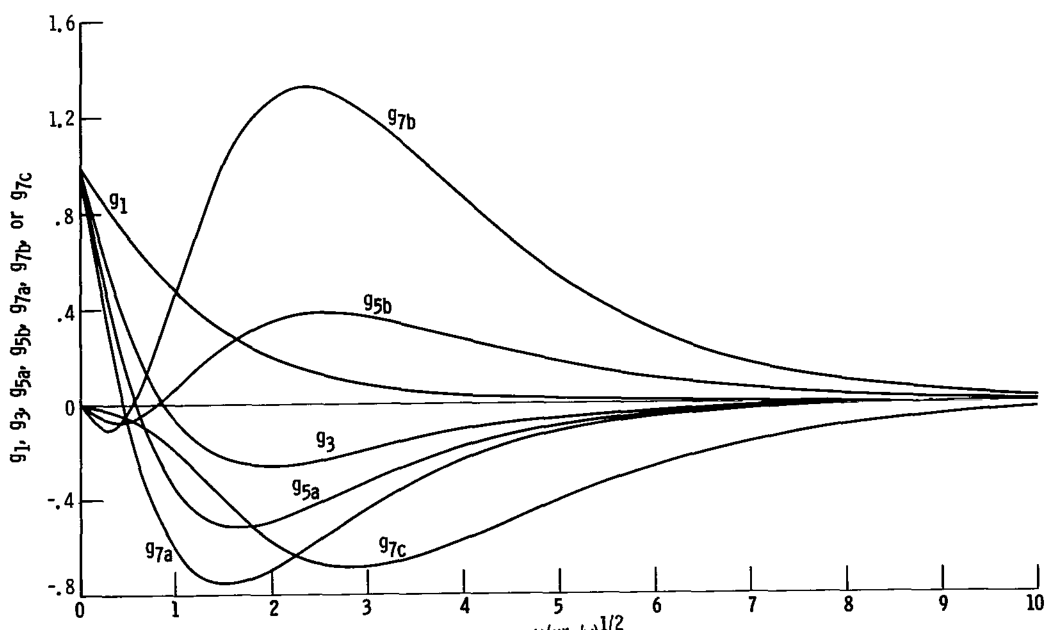


Figure 2. - Series functions related to velocity component in z-direction, w.

$$f_1' \to f_3' \to f_{5a}' \to f_{5b}' \to f_{7a}' \to f_{7b}' \to f_{7c}' \to 0 \quad ; \quad g_1 \to g_3 \to g_{5a} \to g_{5b} \to g_{7a} \to g_{7b} \to g_{7c} \to 0 \quad \Big\} \; \eta \to \infty \tag{6b}$$

In this form, the g- and f'-functions are independent of the r-coefficients and need to be calculated just once for boundary layers on all bodies that can be described by equation (5d). The need for caution in the application of boundary-layer results to bodies with small vertex angles was demonstrated in a rotating cone experiment (ref. 12).

Equations (5) are substituted into equations (4), and the coefficients of the expansion parameters $(r_1X, r_3X^3, \ldots)$ are equated to zero. The successive sets of ordinary differential equations that result are then solved numerically. Curves of the f'- and g-functions, which pertain to the velocity components u and w, are shown in figures 1 and 2. Surface derivatives and limiting f-values are shown in table I.

## TORQUE ON A SPHERE

If a sphere is entirely covered by two hemispherical boundary layers, the torque T exerted on the spinning body is

$$T = 4\pi a^4 \mu \nu^{-1/2} \omega^{3/2} \int_0^{\pi/2} \sin^2\theta \left[\theta g_1'(0) - \frac{1}{6}\theta^3 g_3'(0) + \ldots\right] d\theta \tag{7a}$$

$$= a^4 \mu \nu^{-1/2} \omega^{3/2} [6.709 - 4.081 + 0.744 - 0.100 + \ldots] \tag{7b}$$

$$= 3.27\ a^4 \mu \nu^{-1/2} \omega^{3/2} \tag{7c}$$

where $\mu$ is the absolute viscosity. The successive contributions to the coefficient in equation (7b) are associated with successive odd powers of $\theta$. The uncertainty in the coefficient in equation (7c) probably does not exceed 0.02 because of the rapid convergence of the alternating sequence in equation (7b).

TABLE I. - SURFACE DERIVATIVES AND LIMITING VALUES OF f- and g-FUNCTIONS

| Subscript | f"(0) | g'(0) | f(∞) |
|---|---|---|---|
| 1 | [a]0.510231 | [a]-0.615921 | 0.442 |
| 3 | 1.837947 | -1.485868 | .462 |
| 5a | 2.417578 | -1.992334 | .366 |
| 5b | 1.501100 | -.327129 | .090 |
| 7a | 2.996120 | -2.457972 | .306 |
| 7b | 3.329164 | -.608677 | .146 |
| 7c | .446964 | -.103796 | .001 |

[a]Values to five places have been reported in a study of flow on a rotaing cone (ref. 13).

Bowden and Lord measured coefficients between 3.25 and 4.0, the low values occurring at high speeds and pressures (ref. 9). The agreement between the theoretical and the experimental coefficients is better than is apparent at first glance. When the low coefficients were measured, the experimental conditions indicated that the Reynolds numbers were high and the boundary layers were thin. Under these conditions, the collision region at the equator appears quite small in the schlieren photographs by

Bowden and Lord. This suggests that the collision process made a negligible contribution to the measured torque. Thus, the appropriate comparison for the theoretical torque value, which does not account for the collision process, is provided by the measurements at high Reynolds numbers, namely, the low experimental coefficients. The difference between these low experimental coefficients and the theoretical coefficient is of the order of 1 percent.

In a footnote Bowden and Lord (ref. 9) attribute a coefficient of 3.32 to a recent solution of the momentum integral equation by Howarth and Banks that is more exact than the original solution by Howarth.

Nigam's torque results are

$$T = a^4 \mu \nu^{-1/2} \omega^{3/2} [1.63 + 1.57 + 0.78 + \ldots] \tag{8a}$$

$$= 3.98\ a^4 \mu \nu^{-1/2} \omega^{3/2} \tag{8b}$$

The slow decrease of the successive contributions to the coefficient in equations (8) suggests that their sum may not converge. This questionable convergence is undoubtedly associated with the behavior of Nigam's solution near the equator because the local results near the pole converge rapidly as shown in the section LOCAL SHEAR AND SEPARATION.

## MASS FLUX

In the present solution the question of inflow or outflow from the boundary layer can be settled by the values of $f(\infty)$ in table I because the total mass flux in the layer is

$$\int_0^\infty 2\pi r u \, dy \sim \psi(\infty) \tag{9}$$

On a spherical surface $\psi(\infty)$ increases with $\theta$ up to $\theta \simeq 2\pi/3$, thereby implying that an inflow exists everywhere on an actual sphere except at the equator.

## LOCAL SHEAR AND SEPARATION

On a sphere the shear component $S$ that is developed by the flow toward the equator (illustrated in fig. 3 along with Nigam's results) is

$$S = a\omega^{3/2} \mu \nu^{-1/2} \left[ \theta f_1''(0) - \frac{1}{6} \theta^3 \left( f_3''(0) - f_1''(0) \right) + \ldots \right] \tag{10}$$

The convergence of the present solution seems rapid enough to suggest that the limiting shear curve is between the three- and four-term curves but closer to the four-term curve. Interestingly, the present results show positive shear values at locations beyond the equator that could exist on an experimental sphere if one of the two boundary layers was removed. The fact that separation

($S = 0$) occurs beyond the equator in figure 3 is also suggested by equation (3), which shows that the curvature of $u$ at the surface declines to zero at the equator and reverses for $\theta > \pi/2$. In view of the agreement between the present torque results and experimental results, it appears certain that the local shear values up to the equator in figure 3 are a good approximation to those that exist on an experimental sphere outside the collision zone.

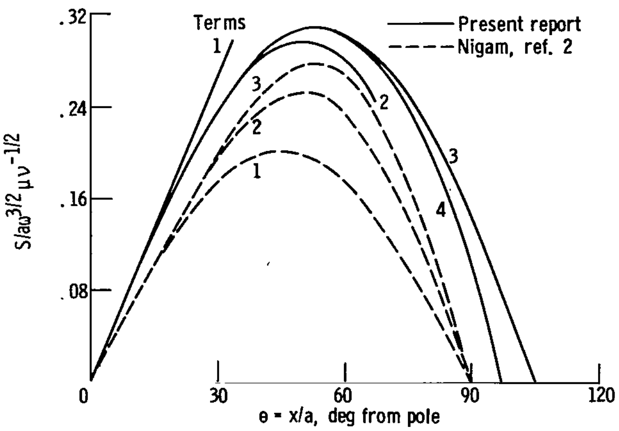


Figure 3. - Shear component on sphere developed by flow toward equator.

Nigam's solution is supposed to satisfy the same boundary-layer equations as the present solution. Neither solution accounts for an equatorial collision. Thus, Nigam's solution is directly comparable to the present solution, but the results in figure 3 deviate grossly at the equator because of Nigam's formulation. This formulation can be justifiably termed erroneous by comparison with the present solution, which is experimentally verified. As a consequence of the erroneous formulation, a divergence in Nigam's series solution is to be expected near the equator, even though the tendency toward such a divergence is not too striking in Nigam's low-order results in figure 3.

## CONCLUSIONS

Nigam's solution to the boundary-layer equations contains an erroneous downstream condition; namely, that the boundary-layer flow away from the pole stops at the equator. Such an equatorial condition arises in an experiment, but it can neither be accommodated by the boundary-layer equations nor does it arise naturally in valid solutions to those equations.

The solution herein yields a torque value for a sphere that agrees with an experimental value. Nigam's theoretical value is higher, and it has questionable convergence; in fact, a local divergence near the equator is to be expected because of the erroneous formulation of Nigam's solution.

In the present solution, which is widely applicable to spinning axisymmetric bodies with conical tips, the surrounding fluid flows into the boundary layer at all locations on a hemisphere. This result agrees with experimental smoke studies that show the only outflow to be at the equator where the boundary layers collide.

Lewis Research Center
National Aeronautics and Space Administration
Cleveland, Ohio, July 20, 1964

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